Idealised GFD Models III

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Revision

So far we have looked at

- The physical properties of fluids represented with equations
- Simplifying equations based on further physical considerations
- Numerical solving differential equations using a fixed time step
- State variables, control parameters and dynamic coupling variables

Field Valued Variables

Consider the vorticity equation for a single ocean layer (eq (26) and (27) from Lecture I)

$$q = \frac{\nabla_H^2 p}{f_0} + \beta(y - y_0) + \frac{f_0}{H}\delta_z(\eta) \quad (1)$$

$$q_t = -(uq)_x - (vq)_y - \frac{f_0}{H}\delta_z(e) + A_2\frac{\nabla_H^4 p}{f_0} \quad (2)$$

The values of u, e, p and q are not just single values, but actually *fields*.

$$q(x,y) = \frac{\nabla_H^2 p(x,y)}{f_0} + \beta(y - y_0) + \frac{f_0}{H}\delta_z(\eta) \quad (3)$$

$$\begin{aligned} q(x,y)_t = & -(u(x,y)q(x,y))_x - (v(x,y)q(x,y))_y \\ & -\frac{f_0}{H}\delta_z(e(x,y)) + A_2\frac{\nabla_H^4 p(x,y)}{f_0} \end{aligned} \quad (4)$$

Spatial Discretisation

- Even on a finite domain, there are an infinite number of points
- We need to simplify our domain to contain a finite number of points for calculations
- The more points, the more accurate the calculations can be.
- The more points, the more calculations are needed per time step.
- Tradeoff between *resolution* and *run time*.

Numerical Grid

The simplest way to choose our points is to place them on a rectangular grid

- A grid translates well to an array of numbers in a programming language
- Spatial derivatives can be quickly computed
- Data can be easily visualised

Numerical differentiation

Consider a field $f(x,y)$ on a rectangular grid, with the value at the grid point (i,j) being $f_{i,j}$. One way to compute df/dx is

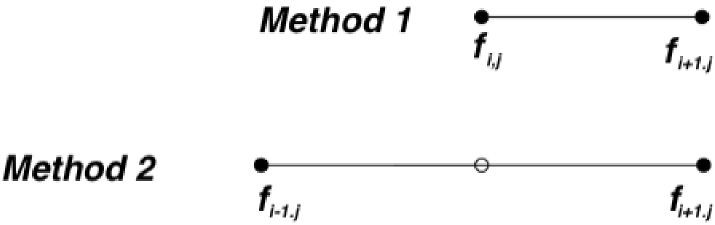
$$\left(\frac{df}{dx}\right)_{i,j} = \frac{f_{i+1,j} - f_{i,j}}{\Delta x} \tag{5}$$

Another way is

$$\left(\frac{df}{dx}\right)_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} \tag{6}$$

Stencils

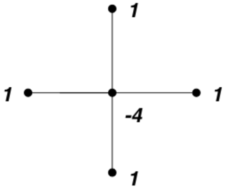
Consider the points used in each of the previous calculations. We call the pattern formed by these points the *stencil*.



Using stencils to discuss algorithms is much easier than keeping track of indices.

Laplacian Stencil

Consider the following stencil



It tells us how to calculate the 5-point laplacian

$$
\begin{aligned}
\nabla^2 f_{i,j} &= \frac{f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j}}{\Delta^2} \qquad (7)\\
&= \left(\frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\Delta x^2}\right) + \left(\frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta y^2}\right) \qquad (8)\\
&= \left(\frac{d^2 f}{dx^2}\right)_{i,j} + \left(\frac{d^2 f}{dy^2}\right)_{i,j} \qquad (9)
\end{aligned}
$$

Sources of Numerical Error

Whenever we do numerical calculations, errors will creep in.

- Physical measurement error
- Floating point error
- Discretisation error

Physical Measurement Error

Any input to the system we derives from a physical measurement will carry an error term.

- Initial state (Temperature, velocity, etc)
- External forcing (Solar radiation, etc)
- Physical parameters (density, heat capacity, etc)

Idealised models are less succeptible to these errors.

Floating Point Error

Computers store numbers using as floating point values. This means they can store a wide range of numbers (up to $10^3 10x$) but only with a finite precision.

- Numbers such as π, $\sqrt(2)$ get truncated, introducing an error.
- Each calculation accumulates previous errors.
- Floating point error typically begins around at 15 or 16 decimal places, but increases with time.

Discretisation error

Calculations such as spatial derivatives rely on approximations to the true value. These are the main source of error in an idealised model

- Errors proportional to Δ or Δ^2, depending on stencil used
- Small Δ decreases errors but requires more computation.
- Making Δ too small can (paradoxically) lead to non-realistic situations.

CFL Condition

If the fluid moves too fast and travels across multiple grid cells in a single time step, our numerical method becomes divergent.

- We compare to u to $\frac{\Delta x}{\Delta t}$.
- If we u is too large (relatively), we say it violates the *CFL condition*
- If we make Δx small to improve numerical accuracy, we must also make Δt small to preserve the CFL condition.

Summary

In the past three lectures we have looked at idealised GFD models. The key aspects of GFD modeling are

- Translating physical processes into mathematical statements.
- Defining state variables, control parameters and coupling variables.
- Using numerical methods to compute discrete values for the state variables.